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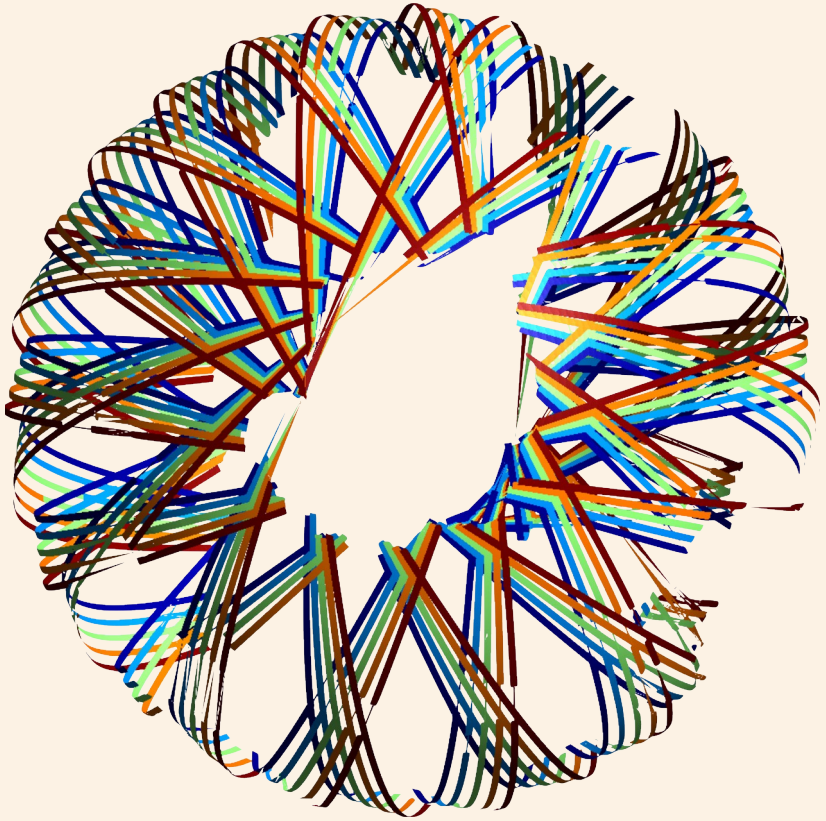
COMEDI

Climate Change Service

Modelling observation error correlations with a regularised diffusion operator

October 29, 2026

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Funded by the European Union

From B to R

In variational DA, we have different constraints for **R** compared to those for **B**:

- For conventional minimisation algorithms, we need $\mathbf{R}^{-1}$, not **R**. (Conversely, these algorithms require **B**, not $\mathbf{B}^{-1}$).

- For generating random samples, we also need **V**, where $\mathbf{R} = \mathbf{V}\mathbf{V}^{T}$. (A similar factorisation is also needed for **B**).

- For diffusion-based correlation operators, we must discretise the operator on a mesh based on observation locations, which changes at each assimilation cycle. (For **B**, the mesh is static).

- We don’t have advanced methods to estimate **R**. (For **B**, we have ensembles).

Correlation operator

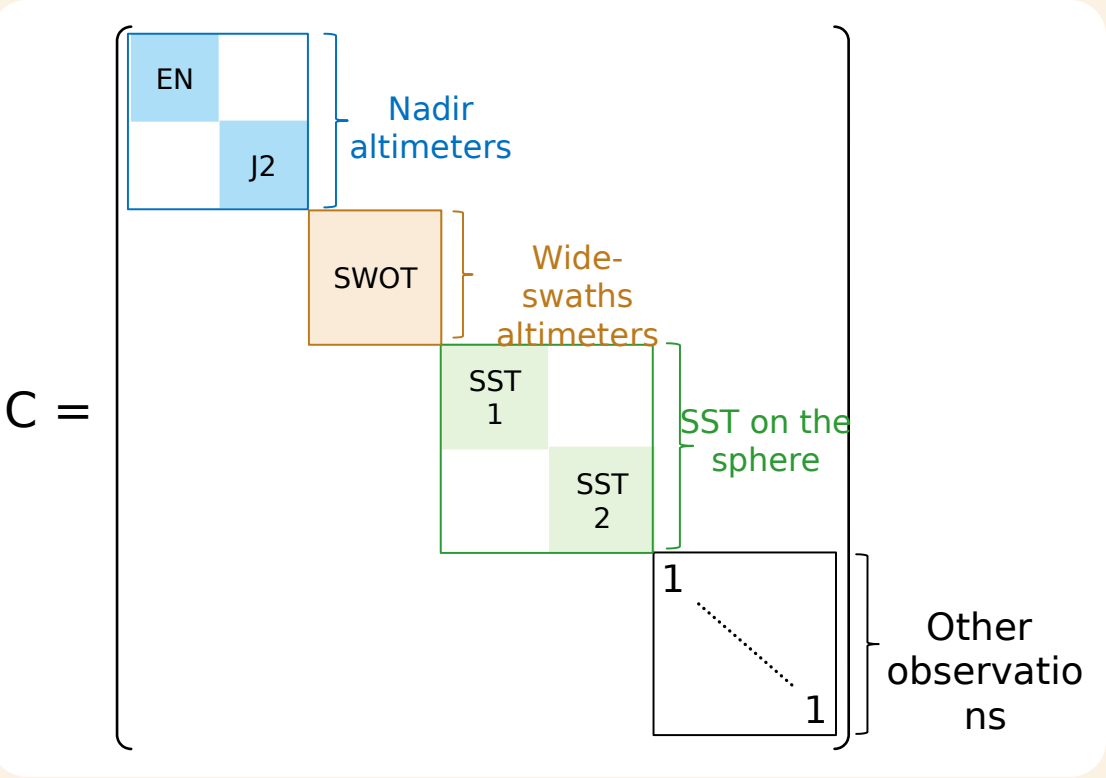
$$\mathbf{R} = \boldsymbol{\Sigma}\,\mathbf{C}\,\boldsymbol{\Sigma}$$

Standard deviations

Correlation operator

For now, we do not account for inter-instrument correlations.

We will focus on a correlation operator for a single instrument.



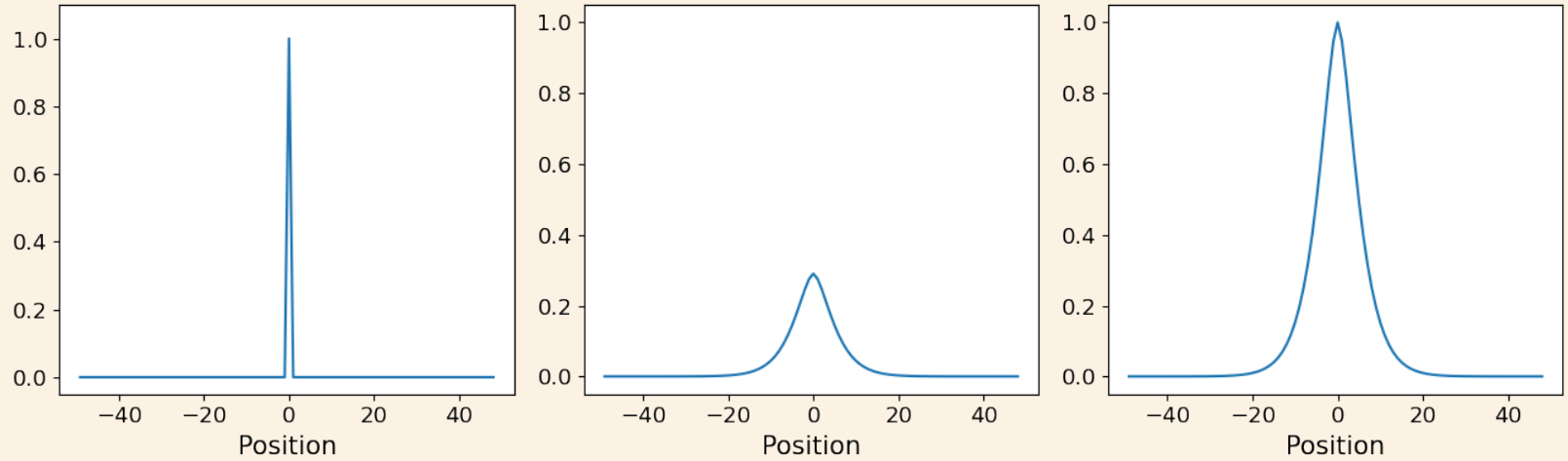
Diffusion-based correlation operators

Correlation operator

Rough

Smooth

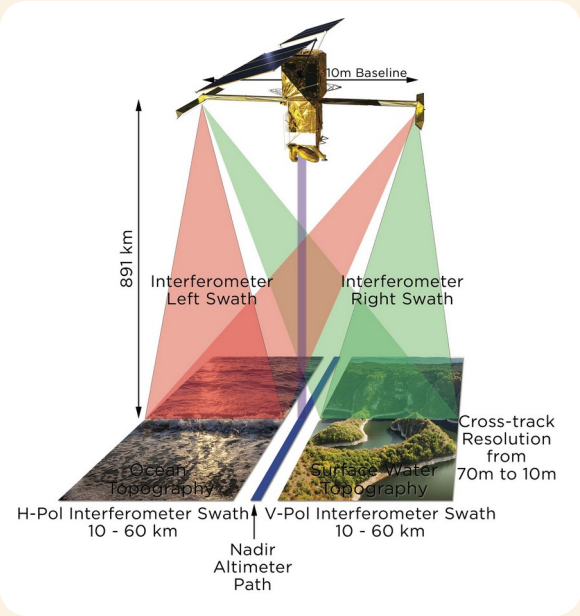
Inverse correlation operator



see Weaver and Mirouze,

Discretisation for wide-swath altimeters

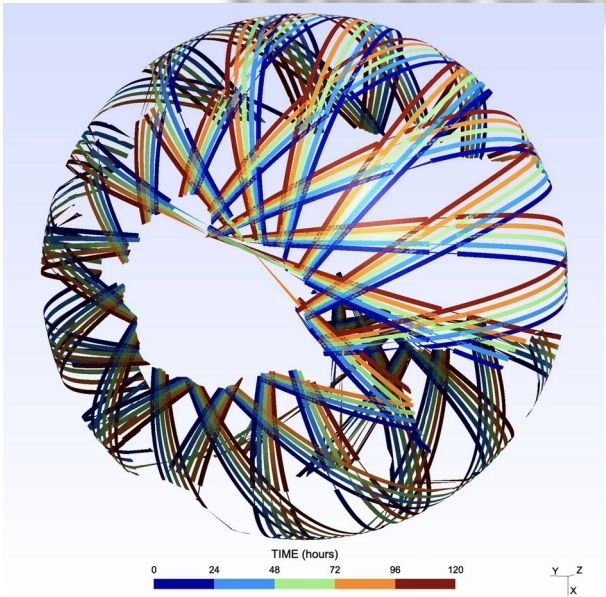
We use data from SWOT (CNES-JPL), the precursor of Sentinel 3-NG (ESA):

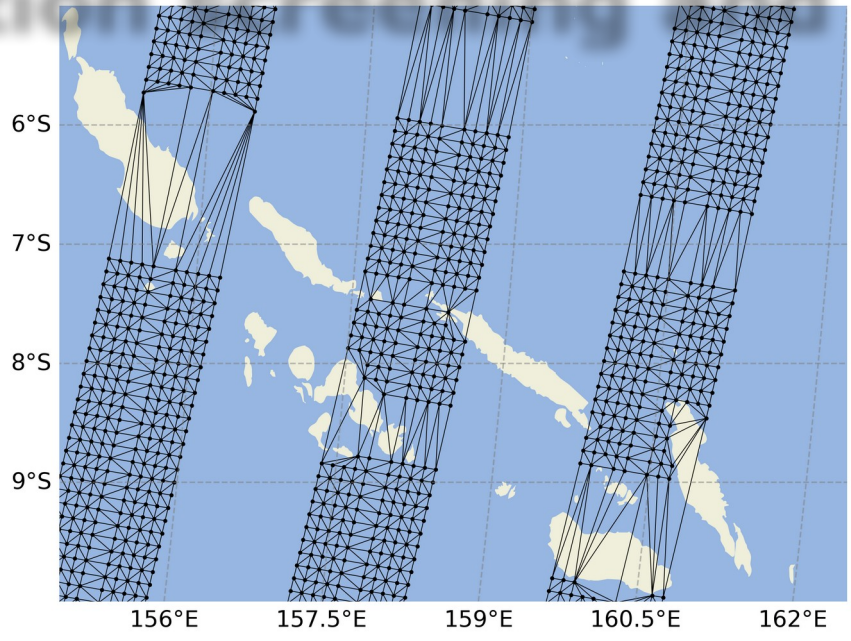




Discretisation for wide-swath altimeters

2D meshes are built with the help of Qhull **after observation screening and quality control.**





Discretisation on a mesh

$$\mathbf{D} = \overline{\mathbf{M}}^{-\frac{1}{2}} \left(\mathbf{S}^{-1}\right)^{\frac{m}{2}} \left(\mathbf{S}^{-\mathrm{T}}\right)^{\frac{m}{2}} \overline{\mathbf{M}}^{-\frac{1}{2}},$$

$$\mathbf{D}^{-1} = \overline{\mathbf{M}}^{\frac{1}{2}} \left(\mathbf{S}\right)^{\frac{m}{2}} \left(\mathbf{S}\right)^{\frac{m}{2}} \overline{\mathbf{M}}^{\frac{1}{2}}$$

with: $\mathbf{S} = \mathbf{I} + \overline{\mathbf{M}}^{-\frac{1}{2}} \mathbf{A} \overline{\mathbf{M}}^{-\frac{1}{2}}$

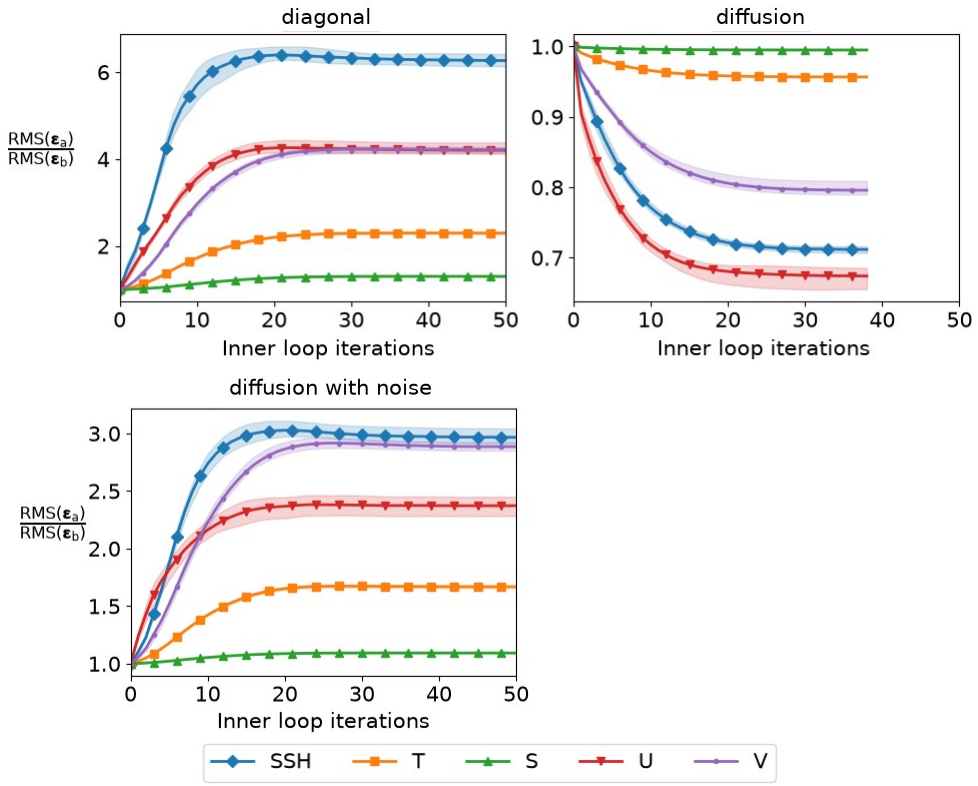
Inverse diffusion step

Stiffness matrix

Mass-lumped mass matrix

See Goux at al. (2025)

Experiments with synthetic SWOT data



Experiments with a simulated innovation vector **d**

$$\mathbf{d} = \mathbf{y} - \mathcal{H}(\mathbf{x}_\mathrm{b})$$
$$\mathbf{d} = \mathcal{H}(\mathbf{x}_\mathrm{t}) + \boldsymbol{\epsilon_o} - \mathcal{H}(\mathbf{x}_\mathrm{t} + \boldsymbol{\epsilon_b})$$
$$\mathbf{d} = \boldsymbol{\epsilon_o} - \mathbf{H}\boldsymbol{\epsilon_b}$$

$$\boldsymbol{\epsilon}_o \sim \mathcal{N}(\mathbf{0}, \mathbf{R})$$
$$\boldsymbol{\epsilon}_b \sim \mathcal{N}(\mathbf{0}, \mathbf{B})$$
$$\boldsymbol{\epsilon}_w \sim \mathcal{N}(\mathbf{0}, \sigma_o^2 \mathbf{I})$$

$$\boldsymbol{\epsilon}_o^{noisy} = \frac{1}{\sqrt{1+\beta^2}}\boldsymbol{\epsilon}_o + \frac{\beta}{\sqrt{1+\beta^2}}\boldsymbol{\epsilon}_w$$

Excessive sensitivity of $\mathbf{R}^{-1}$

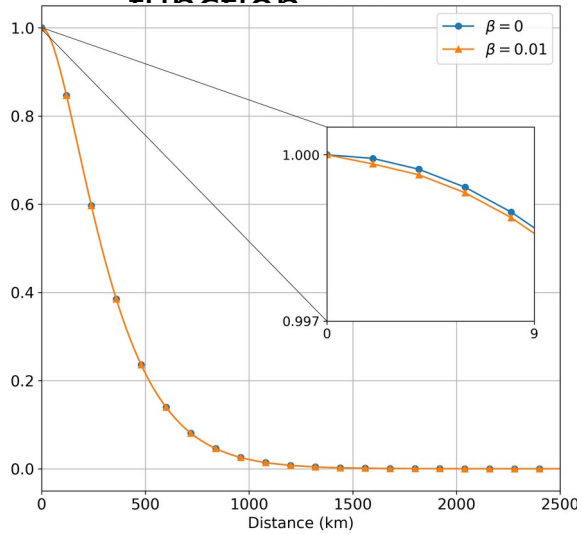
Let's compare the "perfect **R**" for perturbed/unperturbed observations

$$\boldsymbol{\epsilon}_o^{noisy} = \frac{1}{\sqrt{1+\beta^2}}\boldsymbol{\epsilon}_o + \frac{\beta}{\sqrt{1+\beta^2}}\boldsymbol{\epsilon}_w$$

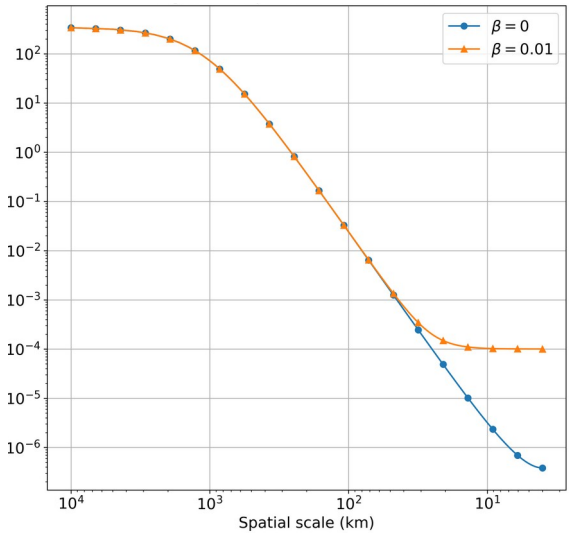
Fitting sample estimates of observation-error correlations to a Matérn function can result in specifying an excessively small error at small spatial scales.

This can lead to an excessive sensitivity of $\mathbf{R}^{-1}$.

Correlation function



Eigenvalue spectrum



Regularised diffusion-based correlation operators

We have developed a regularised formulation of the diffusion-based correlation operator to avoid this problem:

$$\mathbf{D}_\mathrm{r} = \overline{\mathbf{M}}^{-\frac{1}{2}} \left(\mathbf{S}_\mathrm{r}\,\mathbf{S}^{-1}\right)^{\frac{m}{2}} \left(\mathbf{S}^{-\mathrm{T}}\mathbf{S}_\mathrm{r}\right)^{\frac{m}{2}} \overline{\mathbf{M}}^{-\frac{1}{2}},$$

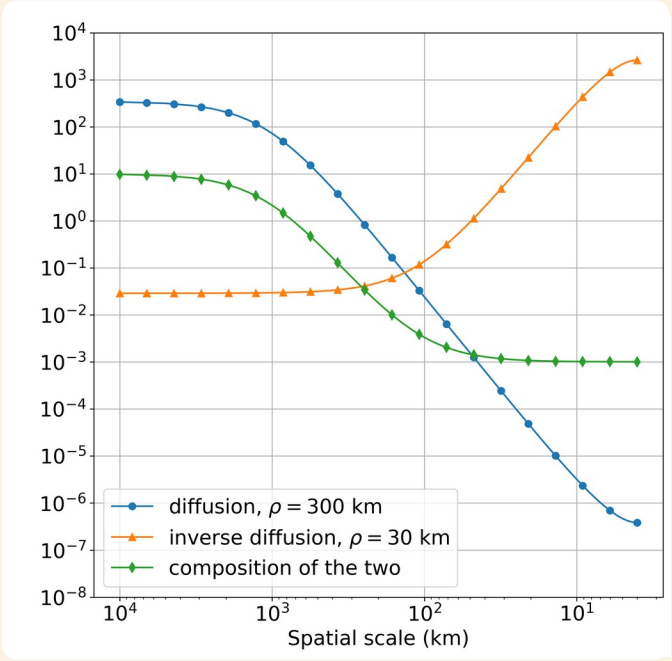
$$\mathbf{D}_\mathrm{r}^{-1} = \overline{\mathbf{M}}^{\frac{1}{2}} \left(\mathbf{S}_\mathrm{r}^{-1}\mathbf{S}\right)^{\frac{m}{2}} \left(\mathbf{S}\,\mathbf{S}_\mathrm{r}^{-\mathrm{T}}\right)^{\frac{m}{2}} \overline{\mathbf{M}}^{\frac{1}{2}}$$

Inverse diffusion step →

$$\mathbf{S} = \mathbf{I} + \overline{\mathbf{M}}^{-\frac{1}{2}} \mathbf{A} \overline{\mathbf{M}}^{-\frac{1}{2}}$$

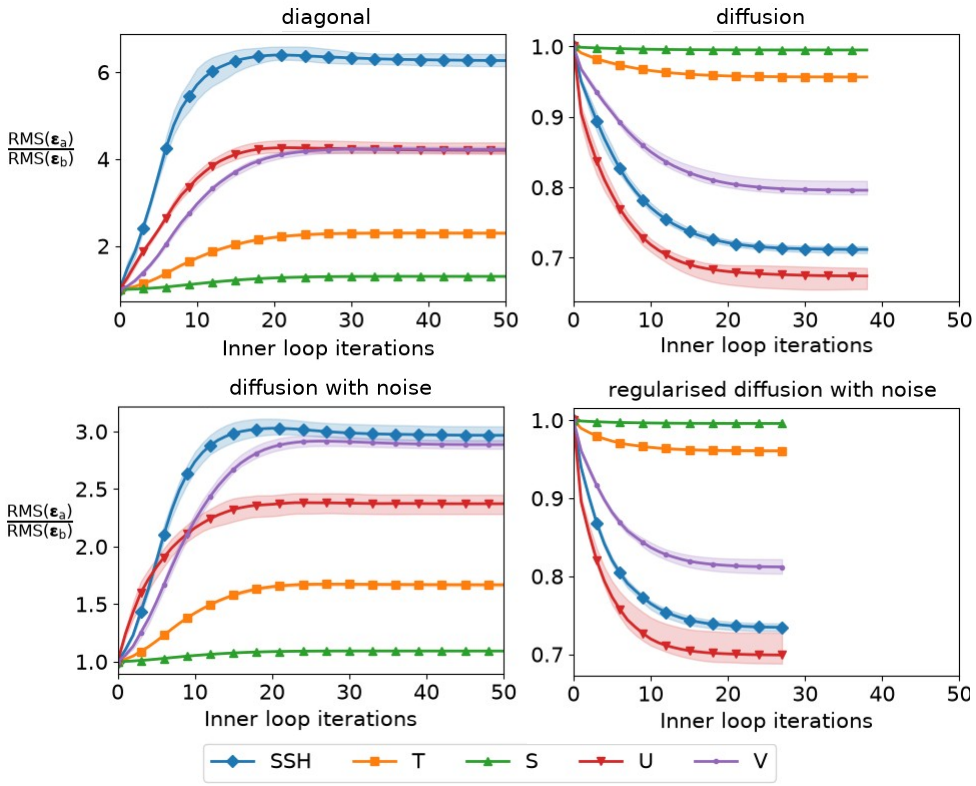
Regularisation step →

$$\mathbf{S}_\mathrm{r} = \mathbf{I} + \mu^2 \overline{\mathbf{M}}^{-\frac{1}{2}} \mathbf{A} \overline{\mathbf{M}}^{-\frac{1}{2}}$$



See Goux at al. (2026)

Experiments with synthetic SWOT data



Experiments with a simulated innovation vector **d**

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$$\boldsymbol{\epsilon}_o^{\,noisy} = \frac{1}{\sqrt{1+\beta^2}}\boldsymbol{\epsilon}_o + \frac{\beta}{\sqrt{1+\beta^2}}\boldsymbol{\epsilon}_w$$

Conclusion

- We have implemented an efficient, parallel observation error correlation operator based on diffusion in NEMOVAR for conventional and wide-swath altimeters as well as daily averaged SST.
- A regularised formulation prevents an excessive sensitivity to small spatial scales of the inverse operator.
- We have derived guidelines on selecting parameters which offers a good compromise between accuracy and numerical performances.

Outlook:

- Evaluation of the impact of the observation error correlations in global and regional ocean applications.
- Implementation of the correlation operator as a model-independent, open-source library
- Acceleration of the estimation of normalisation factors (Graph Neural Networks, Multi-Level Monte Carlo, etc.)

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1. Correlation operators based on implicit diffusion:

   Weaver, A. T. and Mirouze, I. (2013) On the diffusion equation and its application to isotropic and anisotropic correlation modelling in variational assimilation. Quarterly Journal of the Royal Meteorological Society, 139, 242–260. URL: https: //doi.org/10.1002/qj.1955.

2. A practical solver for applying implicit diffusion-based correlation operators:

   Weaver, A. T. Tshimanga and Piacentini, A. (2016). Correlation operators based on an implicitly formulated diffusion equation solved with the Chebyshev iteration. Quarterly Journal of the Royal Meteorological Society, 142, 455–471. URL: https: //doi.org/10.1002/qj.2664.

3. Discretisation on an observation-based mesh:

   Guillet, A., Weaver, A. T., Vasseur, X., Michel, Y., Gratton, S. and Gürol, S. (2019) Modelling spatially correlated observation errors in variational data assimilation using a diffusion operator on an unstructured mesh. Quarterly Journal of the Royal Meteorological Society, 145, 1947–1967.

4. Impact of **R** on convergence in variational DA:

   Goux O, Gürol S, Weaver AT, Diouane Y, Guillet O. Impact of correlated observation errors on the conditioning of variational data assimilation problems. Numer Linear Algebra Appl. 2024; 31(1):e2529.

5. Impact of **R** on accuracy and application to conventional altimetry:

   Goux, O., Weaver, A. T., Gürol, S., Guillet, O. and Piacentini, A. (2025) On the impact of observation-error correlations in data assimilation, with application to along-track altimeter data. Quarterly Journal of the Royal Meteorological Society, 151, e5026.

6. Regularised formulation and application to wide-swath altimetry:

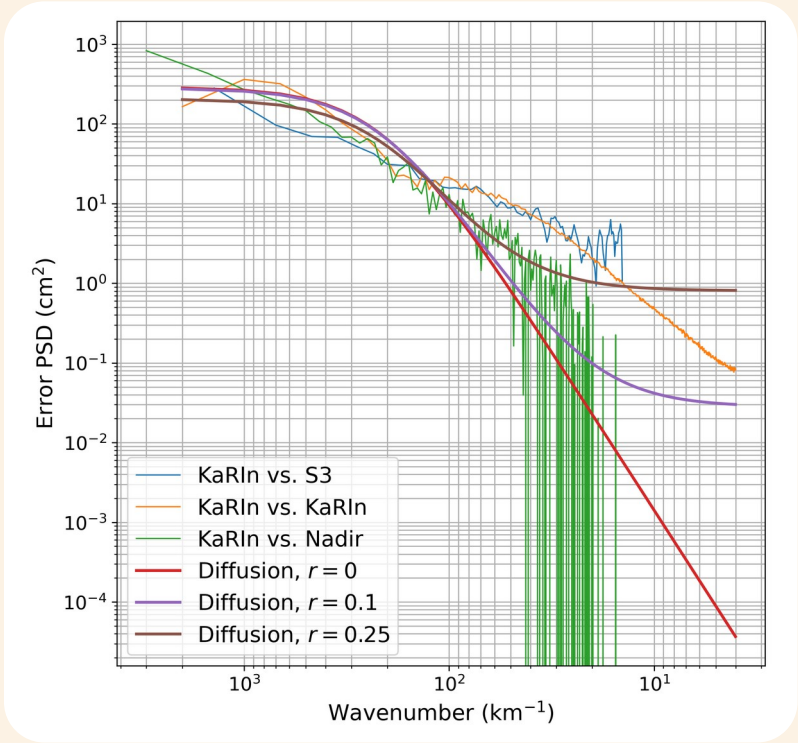
   Goux, O., Weaver, A. T., and Piacentini, A. (2026) Modelling wide-swath altimeter observation-error correlations with a regularised diffusion operator. Manuscript submitted to QJRMS (https://arxiv.org/abs/2610.31409)

Annex: parameters of the operator

- We recommend using **m = 2 diffusion iterations**, as a larger number degrades convergence for little to no added benefit in terms of accuracy.

- With m = 2, the convergence is unlikely to be degraded unless the **observation-error correlation length scale is much larger than the background-error correlation length scale**.

- Observation-error correlation length scales have been estimated using error samples from the CNES-JPL SWOT simulator. They are consistent with currently available diagnostics of SWOT error.

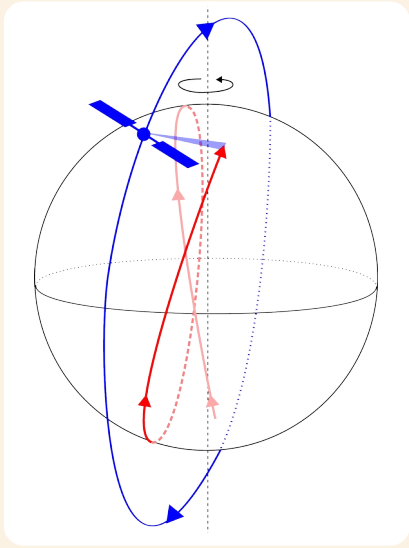


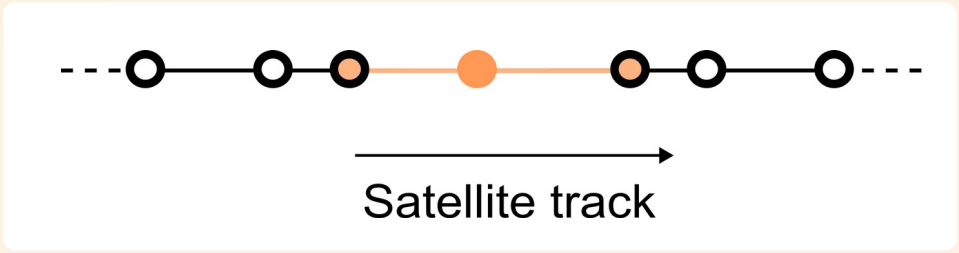
Error estimates from: Nencioli, F., Raynal, M., Ubelmann, C., Cadier, E., Prandi, P. and Dibarboure, G. (2025) *An altimeter-based assessment of swot karin spectral error requirements.* Advances in Space Research, 76, 1241–1261

Annex: Discretisation for conventional altimeters

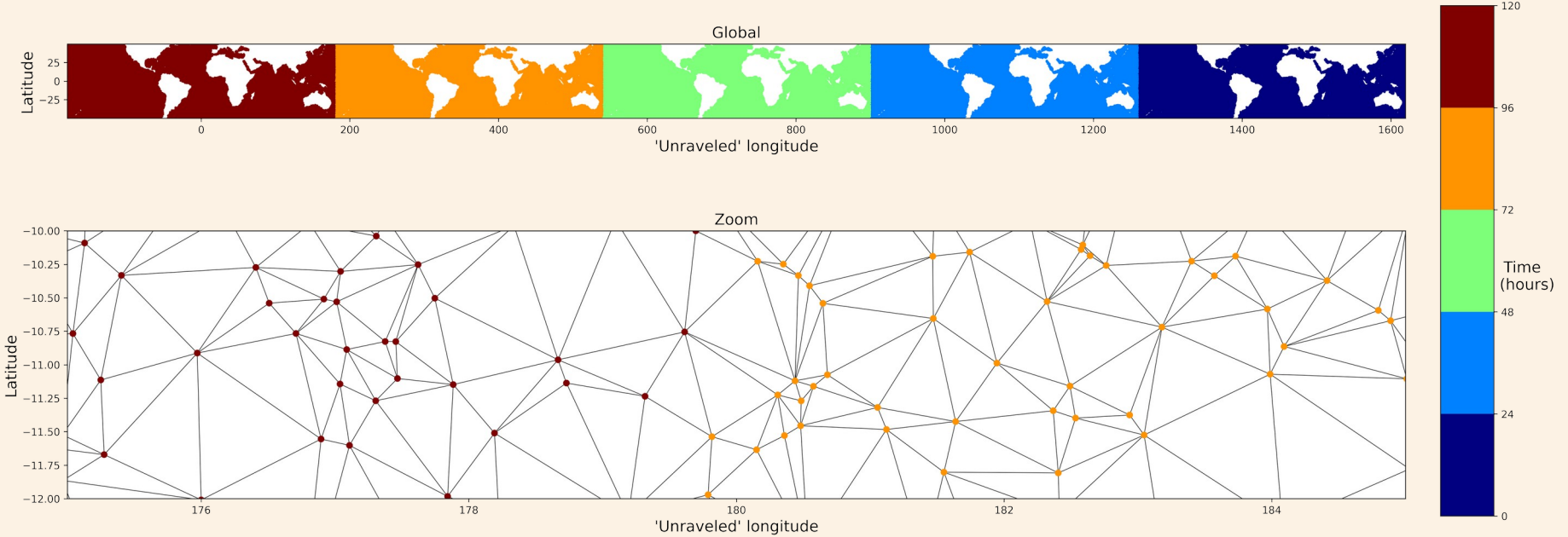
We use Finite-Elements Methods to discretize a diffusion equation on a mesh based on the observation locations

For conventional altimeters, the mesh is one-dimensional.



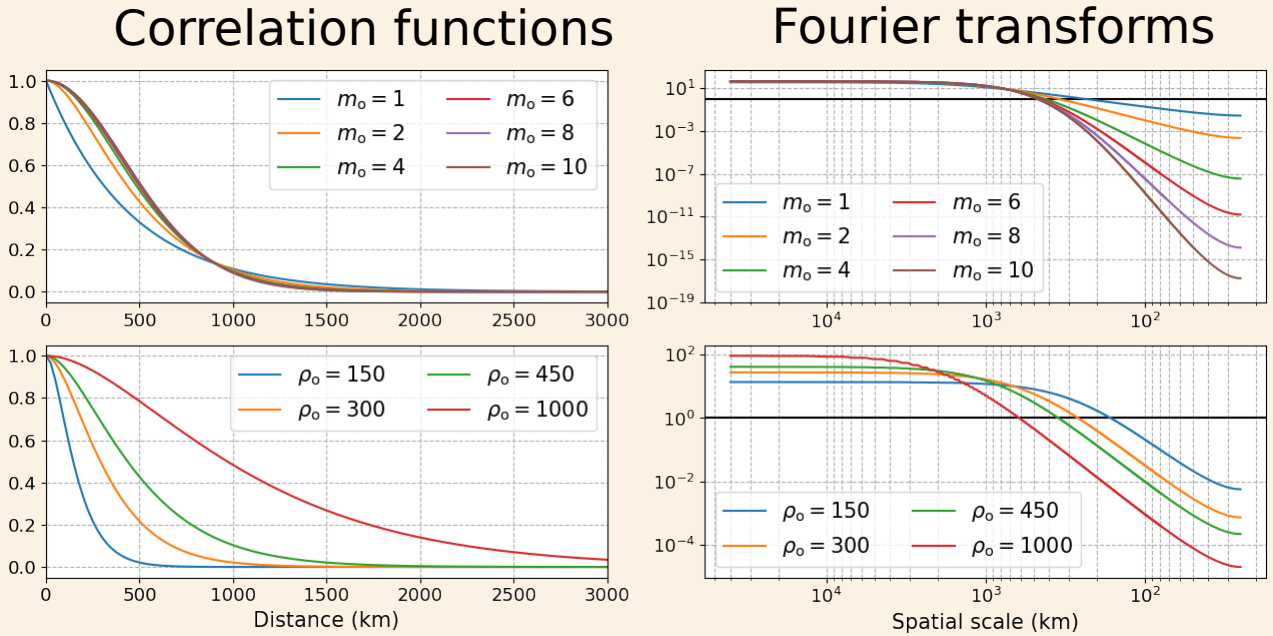


Annex:Discretisation for daily averaged SST



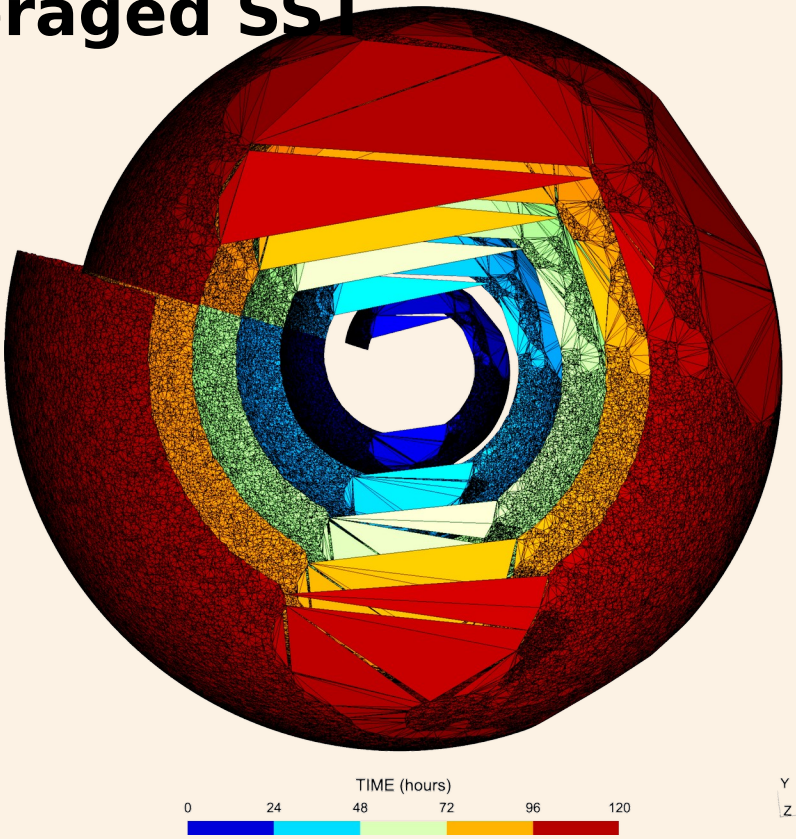
Annex: diffusion-implied correlation functions

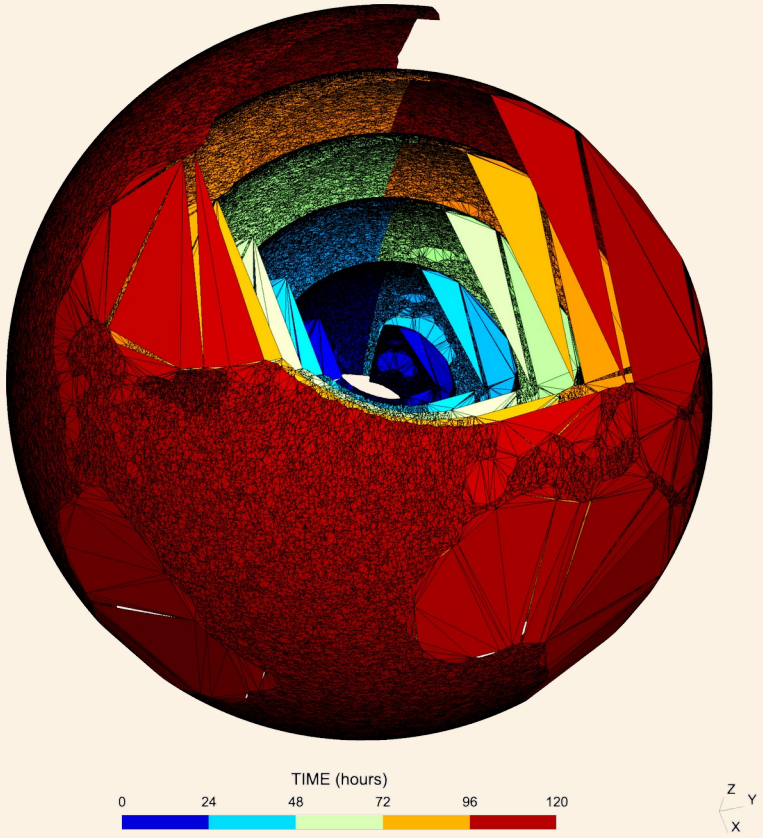
The parameters of the diffusion model can be used to model different correlation functions **from the Matérn family**



see Weaver and Mirouze,

Annex: Discretisation for daily averaged SST





Annex: estimation of the normalisation factors

Normalisation factors have to be estimated at the beginning of each assimilation cycle. They are computed as the inverse of a Monte-Carlo estimator of the standard deviation of the non-normalised operator:

$$\frac{1}{N}\sum_{i=1}^{N}\left(\mathbf{V}_r\,\boldsymbol{\epsilon}_w^{(i)}\cdot\mathbf{V}_r\,\boldsymbol{\epsilon}_w^{(i)}\right)$$

with:

$$\mathbf{V}_{\mathrm{r}}=\overline{\mathbf{M}}^{-\frac{1}{2}}\left(\mathbf{L}^{-1}\right)^{\frac{m}{2}}$$

$$\boldsymbol{\epsilon}_w^{(i)}\sim\mathcal{N}\left(\mathbf{0},\mathbf{I}\right)$$

N = 1000 sample leads to a relative error of 2% on normalisation factors and requires about 50s on 144 CPUs