On the choice of velocity variables for variational ocean data assimilation

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ESA A-TSCV workshop, MOI, Toulouse
Tuesday 13th June 2023

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National Centre for Earth Observation
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Overview

- In NEMOVAR, state variables are transformed to control variables that are assumed to be mutually uncorrelated in the background-error covariance formulation.
- The transformed velocity variables are the *ageostrophic* components.
- However, the components of a horizontal velocity vector are highly correlated, so our “uncorrelated assumption” is incorrect.
- This will result in suboptimal assimilation of future surface current measurements.
- Furthermore, alternative velocity variables that give us better control over horizontal divergence (vertical motions) would be beneficial.

The incremental VAR cost function

Background Error Covariance Matrix

Observation Error Covariance Matrix

$$\mathcal{J}(\delta\mathbf{x}) = \frac{1}{2}\left(\delta\mathbf{x} - (\mathbf{x}^b - \mathbf{x}^{(k)})\right)^T \mathbf{B}^{-1}\left(\delta\mathbf{x} - (\mathbf{x}^b - \mathbf{x}^{(k)})\right) + \frac{1}{2}\left(\mathbf{H}\delta\mathbf{x} - \mathbf{d}\right)^T \mathbf{R}^{-1}\left(\mathbf{H}\delta\mathbf{x} - \mathbf{d}\right)$$

Analysis

Innovation

$$\nabla\mathcal{J}(\delta\mathbf{x}^a) = 0$$

$$\mathbf{d} = \mathbf{y} - h(\mathbf{x}^{(k)})$$

The Control Variable Transform

- **U-Transform** – from control to model space $\quad \delta\mathbf{x} = \mathbf{U}\delta\mathbf{z}$
- **T-Transform** – from model to control space $\quad \delta\mathbf{z} = \mathbf{T}\delta\mathbf{x}$

$$\mathcal{J}(\delta\mathbf{z}) = \frac{1}{2}\left(\delta\mathbf{z} - (\mathbf{z}^b - \mathbf{z}^{(k)})\right)^T \mathbf{U}^T\mathbf{B}^{-1}\mathbf{U}\left(\delta\mathbf{z} - (\mathbf{z}^b - \mathbf{z}^{(k)})\right) + \frac{1}{2}\left(\mathbf{HU}\delta\mathbf{z} - \mathbf{d}\right)^T \mathbf{R}^{-1}\left(\mathbf{HU}\delta\mathbf{z} - \mathbf{d}\right)$$

$$\mathbf{B} = \mathbf{U}\mathbf{U}^T$$

$$\mathcal{J}(\delta\mathbf{z}) = \frac{1}{2}\left(\delta\mathbf{z} - (\mathbf{z}^b - \mathbf{z}^{(k)})\right)^T \left(\delta\mathbf{z} - (\mathbf{z}^b - \mathbf{z}^{(k)})\right) + \frac{1}{2}\left(\mathbf{HU}\delta\mathbf{z} - \mathbf{d}\right)^T \mathbf{R}^{-1}\left(\mathbf{HU}\delta\mathbf{z} - \mathbf{d}\right)$$

New cost function that no longer requires us to find the inverse of B

CVT in the ocean DA

- **Balance operator** is used to describe balance relationships,

$$\mathbf{B} = \mathbf{K}\mathbf{\Sigma}\mathbf{C}\mathbf{\Sigma}\mathbf{K}^T \qquad \mathbf{U} = \mathbf{K}\mathbf{\Sigma}\mathbf{C}^{\frac{1}{2}}$$

$$\mathbf{v} = \mathbf{v}_B + \mathbf{v}_U$$

Balanced Unbalanced

- **Unbalanced** (ageostrophic) components of the velocities are used as control variables.

A common CVT in atmospheric DA

- Use **Helmholtz theorem.**

$$\mathbf{v} = \mathbf{v}_\psi + \mathbf{v}_\chi$$

- Split velocities in rotational and divergent parts:
  stream function and **velocity potential.**

- The assumption that their errors are uncorrelated is more appropriate.
- Stream function and velocity potential are used as control variables.

A revised CVT in ocean DA

- **Objective:** Apply Helmholtz theorem to the unbalanced horizontal velocity vectors.

- Unbalanced stream function and velocity potential will be used as control variables.

$$(u_U, v_U) \longrightarrow (\psi_U, \chi_U)$$

A revised CVT in ocean DA

- Difficulty in the ocean arises when performing the T-transform.

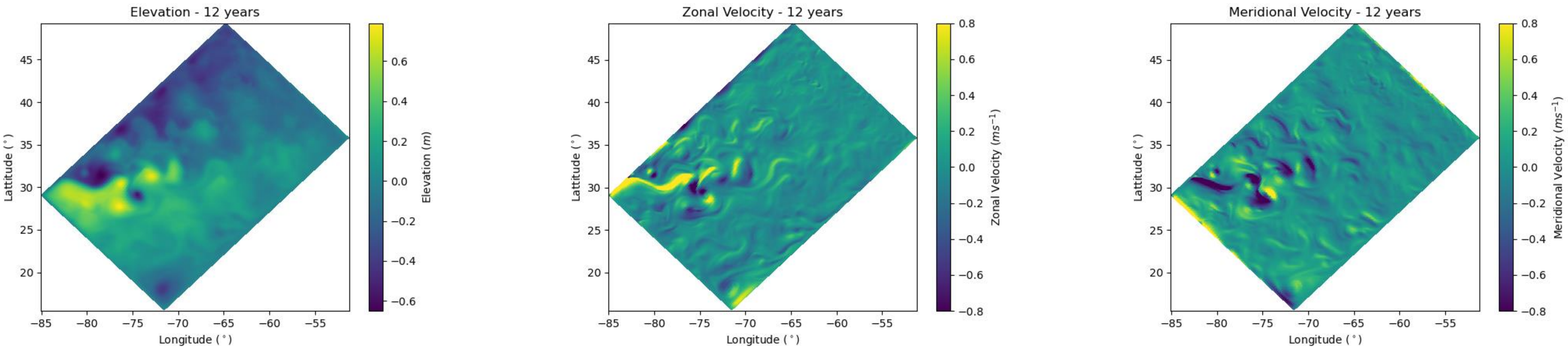
$$\delta \mathbf{z} = \mathbf{T}\delta\mathbf{x}$$

$$(u_U, v_U) \longrightarrow (\psi_U, \chi_U)$$

- T-Transform is essential to compute statistics.
- Must solve an elliptic equation with appropriate boundary conditions.
- Li et al. (2006) proposed using Tikhonov's regularisation for a unique solution.

Future Work

- Run correlation analysis using the SWES and Gyre Configuration in NEMO.
- Run assimilation experiments with these control variables in NEMOVAR.



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Thank you!

Any questions?

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Shallow Water Equations

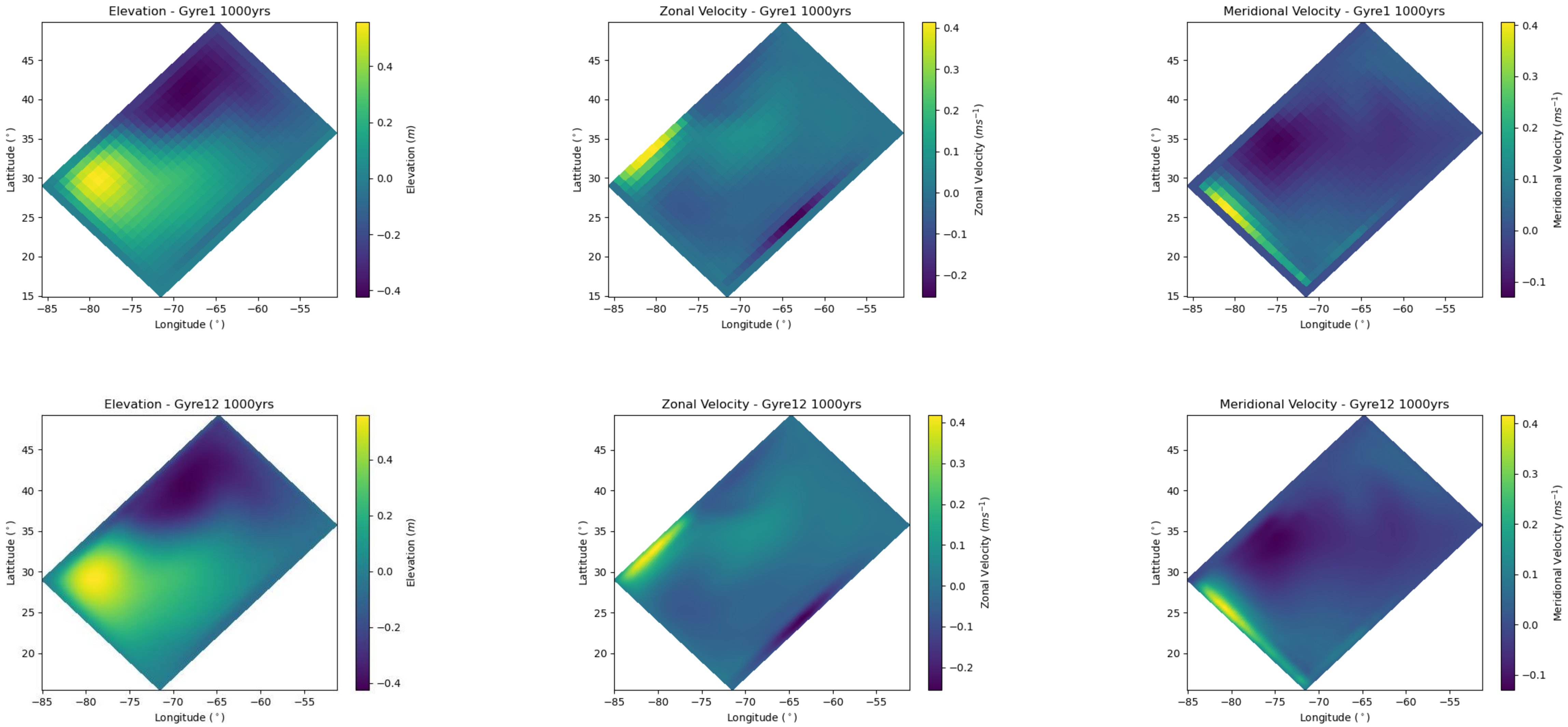
- SWEs on a beta plane.
- We will take elevation in its totality.
- Control variables:
  - Elevation,
  - Unbalanced components of stream function and velocity potential.

$$\delta \mathbf{x} = \begin{pmatrix} \delta\eta \\ \delta\mathbf{u} \\ \delta\mathbf{v} \end{pmatrix} \longrightarrow \delta \mathbf{z} = \begin{pmatrix} \delta\eta \\ \delta\psi_u \\ \delta\chi_u \end{pmatrix}$$

Gyre Configuration

- Simulate seasonal cycle of a double-gyre box model
- Analytical seasonal forcing
- Spontaneous generation of interacting, transient mesoscale eddies
- Beta plane
- Bounded by vertical walls and a flat bottom
- Idealised north Atlantic basin
- Initiated at rest
- Vertical profiles of temperature and salinity uniformly applied to the whole domain

Gyre Configuration



T-transform

$$\delta \mathbf{x} = \mathbf{U}\delta \mathbf{z}$$

$$\delta \mathbf{z} = \begin{pmatrix} \delta\eta \\ \psi_u \\ \chi_u \end{pmatrix} \text{ and } \mathbf{x} = \begin{pmatrix} \delta\eta \\ \delta\mathbf{u} \\ \delta\mathbf{v} \end{pmatrix}$$

1. Find the balanced velocities, $\delta \mathbf{u}_b = -\frac{g}{f}\frac{\partial \delta\eta}{\partial y}$ and $\delta \mathbf{v}_b = \frac{g}{f}\frac{\partial \delta\eta}{\partial x}$
2. Find the unbalanced velocities using,

$$\delta \mathbf{u}_u = \delta \mathbf{u} - \delta \mathbf{u}_b,$$

$$\delta \mathbf{v}_u = \delta \mathbf{v} - \delta \mathbf{v}_b.$$

3. Find the unbalanced velocity potential, χ_u and unbalanced streamfunction, ψ_u, from $\delta \mathbf{u}_u$ and $\delta \mathbf{v}_u$

$$\delta \mathbf{u}_u = -\frac{\partial \psi_u}{\partial y} + \frac{\partial \chi_u}{\partial x}$$

$$\delta \mathbf{v}_u = \frac{\partial \psi_u}{\partial x} + \frac{\partial \chi_u}{\partial y}$$

4. Store the spatial means of $\delta \mathbf{u}$ and $\delta \mathbf{v}$ i.e. $< \delta \mathbf{u} >, < \delta \mathbf{v} >$.